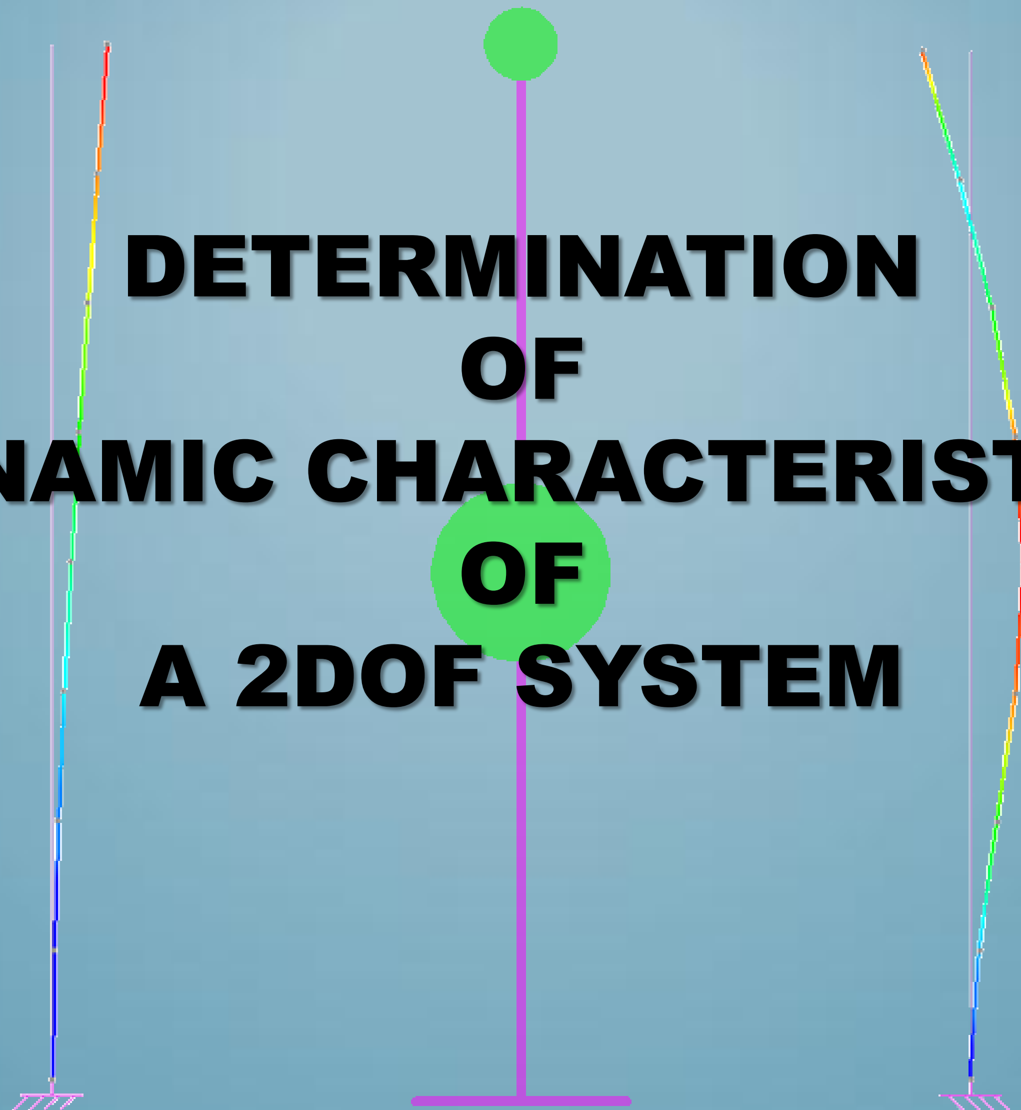
DETERMINATION OF DYNAMIC CHARACTERISTICS OF A 2DOF SYSTEM



by Zoran VARGA, Ms.C.E.

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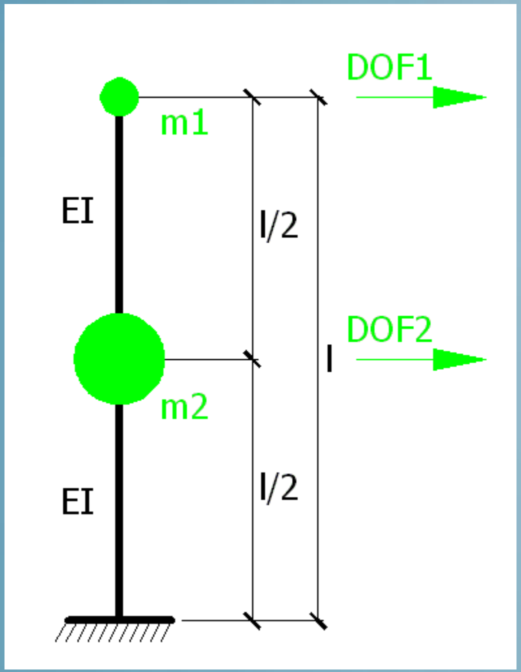
- ***The 2 DOF System***



m_1=3m		[kg]
m_2=8m	m=10	[kg]
l=2		[m]
E=210000		[N/mm²]
I=19		[cm⁴]

- <u>*Symbols*</u>

DOF – Degree of Freedom
m_i – mass at level i
j – Degree of Freedom
[Δ] – Flexibility Matrix
[K] – Stiffness Matrix
[M] – Mass Matrix
[I] – Identity Matrix
A – Area of the Moment Diagram
y – the ordinate along the center of gravity of the moment diagram
ω – the pulse of the system
T – the period of the system
f – the frequency of the system
k – vibration mode
φ_{jk} – Eigenvector
[Φ] – Mode Shape Matrix
L_k – Modal Participation Factor
M_k – Modal Mass
$m_{eff,k}$ – Effective Modal Mass

In order to determinate the dynamic characteristics 2 methods were applied:

- **FLEXIBILITY** Method
- **STIFFNESS** Method

- ***The FLEXIBILITY Method***

- *The general equation of the Eigen vibrations of a vibrating system using the Flexibility Matrix [Δ]:*

$$|[\boldsymbol{\Delta}] \cdot [\boldsymbol{M}] - \boldsymbol{\lambda}[\boldsymbol{I}]| \cdot \{\boldsymbol{\phi}_{jk}\} = \mathbf{0} \qquad (1)$$

- *In order to have non-zero solutions for the equation the determinant must then be zero:*

$$|[\boldsymbol{\Delta}] \cdot [\boldsymbol{M}] - \boldsymbol{\lambda}[\boldsymbol{I}]| = \mathbf{0} \qquad (2)$$

where $\lambda = \frac{1}{\omega^2}$

[Δ] – Flexibility Matrix

[M] – Mass Matrix

[I] – Identity Matrix

- *The Mass Matrix*

$$[\mathrm{M}] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad [\mathrm{M}] = m \cdot \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix} \qquad (3)$$

- ***The FLEXIBILITY Method***
  - ***Determination of the FLEXIBILITY Matrix***

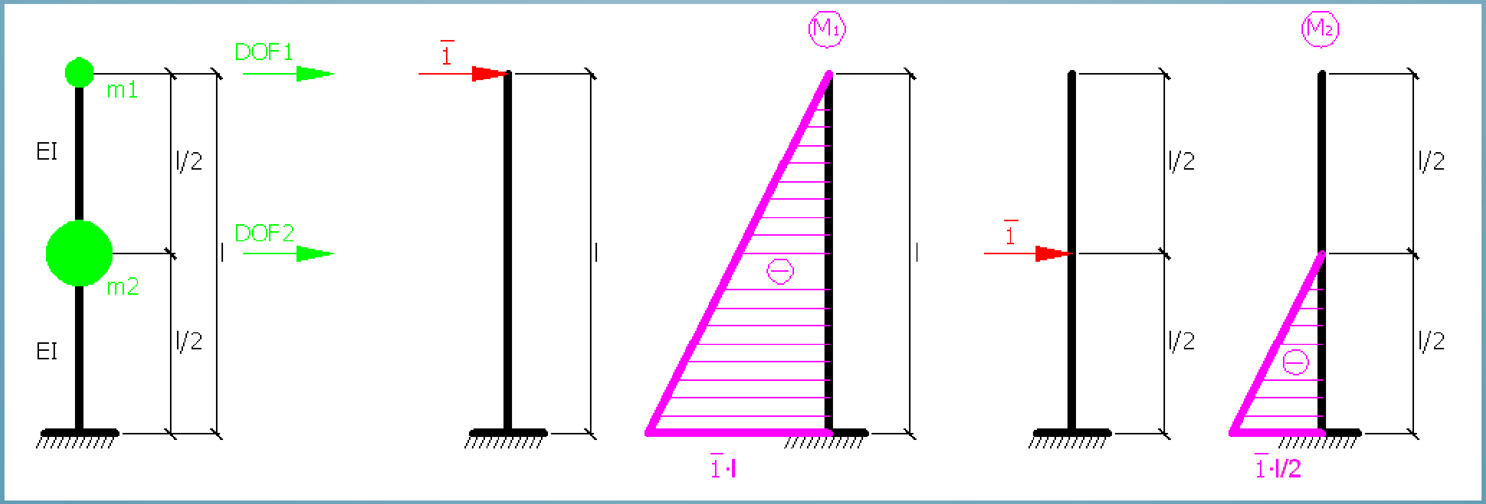
$$[\Delta] = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} \qquad (4)$$

δ_{ij} – the displacement of the mass i in the j direction

➢ *Mohr – Maxwell Formula:*

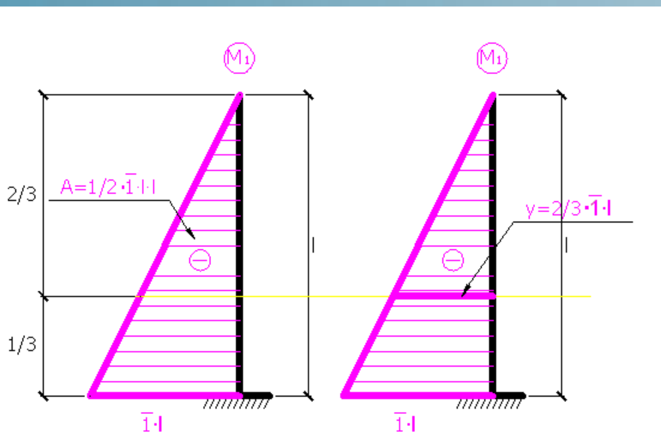
$$\delta_{ij} = \sum \int \frac{M_i(x) \cdot M_j(x)}{EI} dx = \sum \frac{A_i \cdot y_j}{EI} \qquad (5)$$

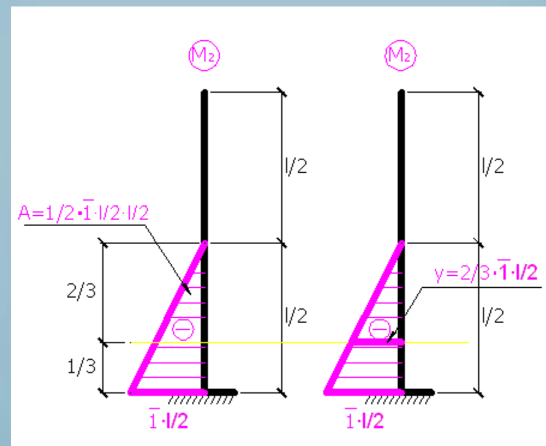
➢ *Moment diagrams obtained from applying the unit force in the direction of the DOF*

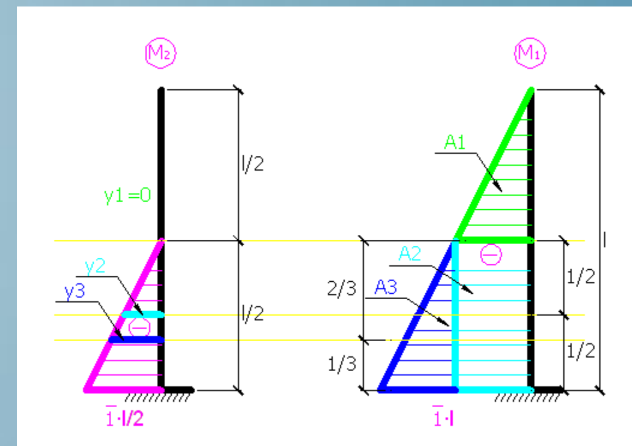


• The FLEXIBILITY Method

• Determination of the FLEXIBILITY Matrix







The displacement of mass m_1 in the 1^{st}DOF direction:

$$\delta_{11} = \frac{1}{EI} \cdot A \cdot y$$

$$\delta_{11} = \frac{1}{EI} \cdot \left(\frac{1}{2} \cdot \bar{1} \cdot l \cdot l\right) \cdot \left(\frac{2}{3} \cdot \bar{1} \cdot l\right)$$

$$\boldsymbol{\delta_{11} = \frac{l^3}{3EI}}$$

The displacement of mass m_2 in the 2^{nd}DOF direction:

$$\delta_{22} = \frac{1}{EI} \cdot A \cdot y$$

$$\delta_{22} = \frac{1}{EI} \cdot \left(\frac{1}{2} \cdot \bar{1} \cdot \frac{l}{2} \cdot \frac{l}{2}\right) \cdot \left(\frac{2}{3} \cdot \bar{1} \cdot \frac{l}{2}\right)$$

$$\boldsymbol{\delta_{22} = \frac{l^3}{24EI}}$$

The displacement of mass m_1 in the 2^{nd}DOF direction:

$$\delta_{12} = \delta_{21}$$

(Theorem of reciprocal unit displacements)

$$\delta_{12} = \frac{1}{EI} \cdot (A_1 \cdot y_1 + A_2 \cdot y_2 + A_3 \cdot y_3)$$

$$\delta_{12} = \frac{1}{EI} \cdot \left[\left(\frac{1}{2} \cdot \bar{1} \cdot \frac{l}{2} \cdot \frac{l}{2}\right) \cdot 0 + \left(\bar{1} \cdot \frac{l}{2} \cdot \frac{l}{2}\right) \cdot \left(\frac{1}{2} \cdot \bar{1} \cdot \frac{l}{2}\right) + \left(\frac{1}{2} \cdot \bar{1} \cdot \frac{l}{2} \cdot \frac{l}{2}\right) \cdot \left(\frac{2}{3} \cdot \bar{1} \cdot \frac{l}{2}\right)\right]$$

$$\boldsymbol{\delta_{12} = \delta_{21} = \frac{5l^3}{48EI}}$$

$$[\Delta] = \frac{l^3}{48EI} \cdot \begin{bmatrix} 16 & 5 \\ 5 & 2 \end{bmatrix}$$

- ## *The STIFFNESS Method*

- *The general equation of the Eigen vibrations of a vibrating system using the Stiffness Matrix [K]:*

$$\left|[\boldsymbol{K}] - \boldsymbol{\omega}^{2} \cdot [\boldsymbol{M}]\right| \cdot \left\{\boldsymbol{\phi}_{jk}\right\} = \mathbf{0} \qquad (6)$$

- *In order to have non-zero solutions for the equation the determinant must then be zero:*

$$\left|[\boldsymbol{K}] - \boldsymbol{\omega}^{2} \cdot [\boldsymbol{M}]\right| = \mathbf{0} \qquad (7)$$

[K] – Stiffness Matrix

[M] – Mass Matrix

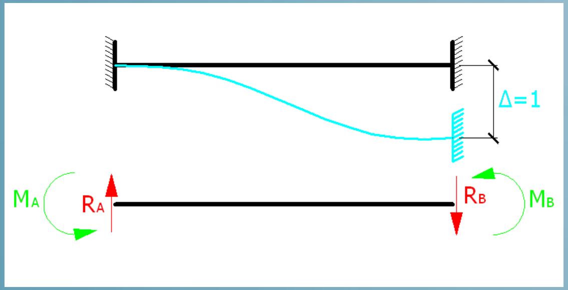
- *The Mass Matrix*

$$[\mathrm{M}] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad [\mathrm{M}] = m \cdot \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$$

- ***The STIFFNESS Method***
  - ***Determination of the STIFFNESS Matrix***

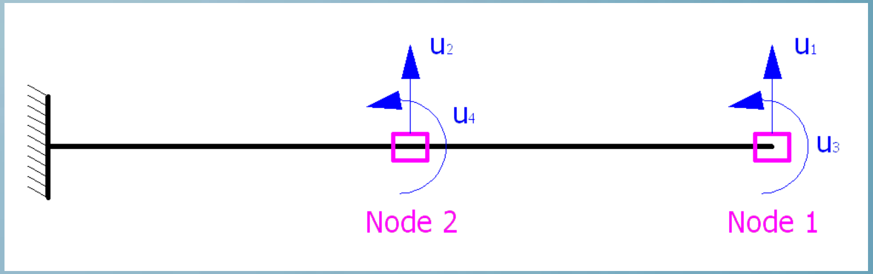
$$\boldsymbol{k} = \frac{\boldsymbol{R}}{\Delta} = \frac{\boldsymbol{M}}{\boldsymbol{\varphi}} \qquad (8)$$

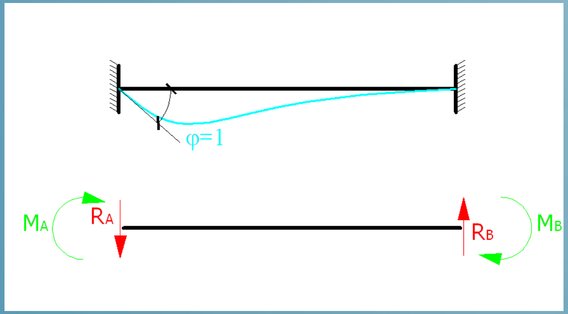
Loading conditions



$$M_A = M_B = \frac{6EI}{L^2}$$

$$R_A = R_B = \frac{12EI}{L^3}$$





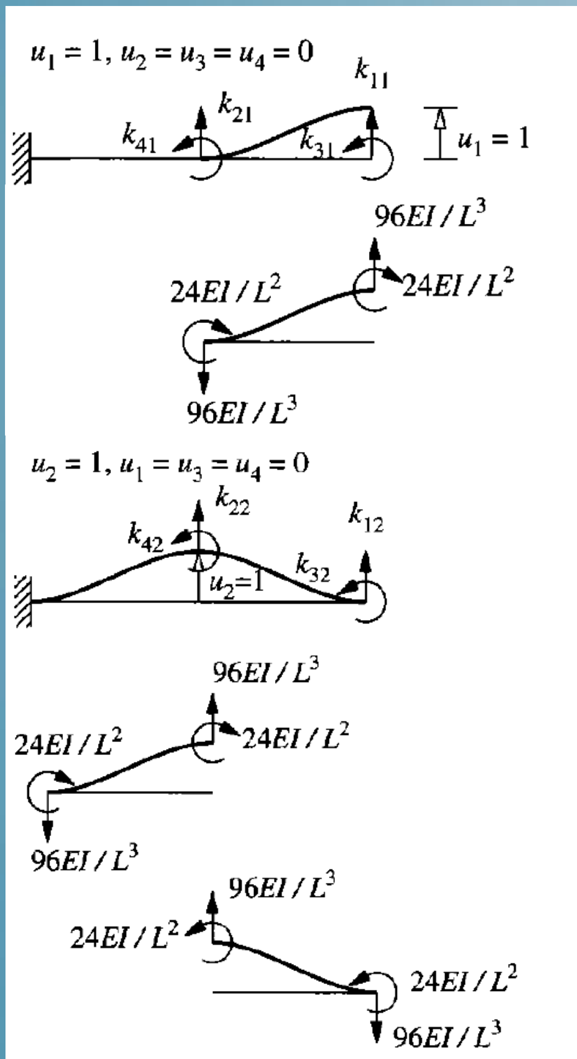
$$M_A = \frac{4EI}{L}$$

$$M_B = \frac{2EI}{L}$$

$$R_A = R_B = \frac{6EI}{L^2}$$

$$[\boldsymbol{K}] = \begin{bmatrix} \boldsymbol{k_{11}} & \boldsymbol{k_{12}} & \boldsymbol{k_{13}} & \boldsymbol{k_{14}} \\ \boldsymbol{k_{21}} & \boldsymbol{k_{22}} & \boldsymbol{k_{23}} & \boldsymbol{k_{24}} \\ \boldsymbol{k_{31}} & \boldsymbol{k_{32}} & \boldsymbol{k_{33}} & \boldsymbol{k_{34}} \\ \boldsymbol{k_{41}} & \boldsymbol{k_{42}} & \boldsymbol{k_{43}} & \boldsymbol{k_{44}} \end{bmatrix} \qquad (9)$$

- ## The STIFFNESS Method
    - *Determination of the STIFFNESS Matrix*



$$k_{11} = -k_{21} = \frac{12EI}{L^3} = \frac{12EI}{(l/2)^3} = \frac{96EI}{l^3}$$

$$k_{31} = k_{41} = -\frac{6EI}{L^2} = -\frac{6EI}{(l/2)^2} = \frac{-24EI}{l^2}$$

$$k_{13} = -k_{23} = \frac{6EI}{L^2} = \frac{6EI}{(l/2)^2} = \frac{-24EI}{l^2}$$

$$k_{33} = \frac{4EI}{L} = \frac{4EI}{(l/2)} = \frac{8EI}{l}$$

$$k_{43} = \frac{2EI}{L} = \frac{2EI}{(l/2)} = \frac{4EI}{l}$$

$$k_{12} = \frac{-96EI}{l^3}$$

$$k_{22} = \frac{96EI}{l^3} + \frac{96EI}{l^3} = \frac{192EI}{l^3}$$

$$k_{32} = \frac{24EI}{l^2}$$

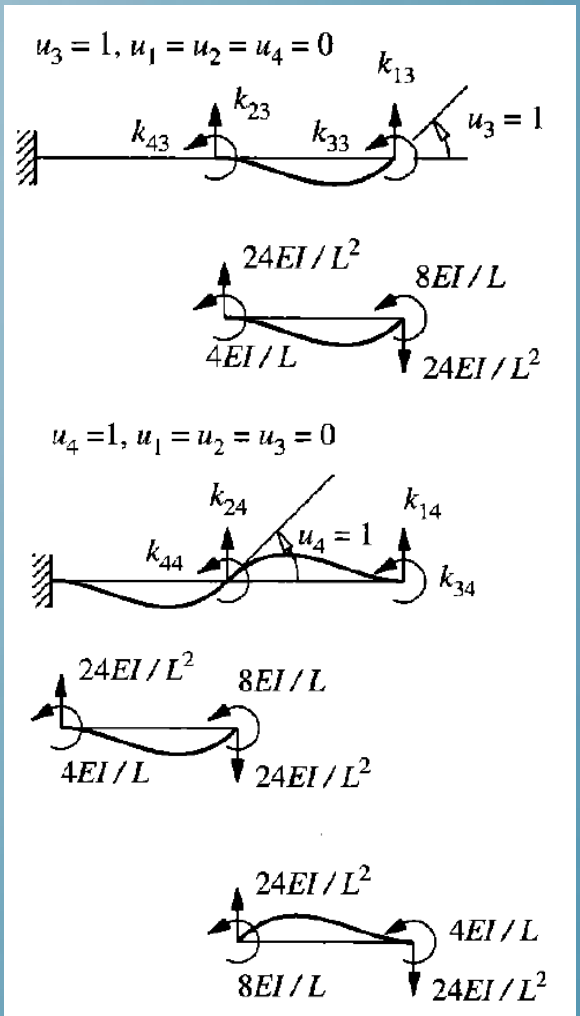
$$k_{42} = \frac{24EI}{l^2} - \frac{24EI}{l^2} = 0$$

$$k_{14} = \frac{-24EI}{l^2}$$

$$k_{24} = 0$$

$$k_{34} = \frac{4EI}{l}$$

$$k_{44} = \frac{16EI}{l}$$



- **The STIFFNESS Method**
  - ***Determination of the STIFFNESS Matrix***

$$[K] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

$$[\boldsymbol{K}] = \frac{\boldsymbol{8EI}}{\boldsymbol{l^3}} \cdot \left[\begin{array}{cc|cc} \mathbf{12} & \mathbf{-12} & \mathbf{-3}\boldsymbol{l} & \mathbf{-3}\boldsymbol{l} \\ \mathbf{-12} & \mathbf{24} & \mathbf{3}\boldsymbol{l} & \mathbf{0} \\ \hline \mathbf{-3}\boldsymbol{l} & \mathbf{3}\boldsymbol{l} & \boldsymbol{l^2} & \boldsymbol{l^2}/\mathbf{2} \\ \mathbf{-3}\boldsymbol{l} & \mathbf{0} & \boldsymbol{l^2}/\mathbf{2} & \mathbf{2}\boldsymbol{l^2} \end{array}\right] = \begin{bmatrix} k_{tt} & k_{t0} \\ k_{0t} & k_{00} \end{bmatrix} \qquad (10)$$

➢ ***Condensed STIFFNESS Matrix:*** $\mathrm{K} = k_{tt} - {k_{0t}}^{T} \cdot {k_{00}}^{-1} \cdot k_{0t}$ *(11)*

$$[\boldsymbol{K}] = \frac{\boldsymbol{48EI}}{\boldsymbol{7l^3}} \cdot \begin{bmatrix} \mathbf{2} & \mathbf{-5} \\ \mathbf{-5} & \mathbf{16} \end{bmatrix}$$

- ***The Dynamic Characteristics***

The FLEXIBILITY Method

$$\left|[\Delta]\cdot[\boldsymbol{M}]-\boldsymbol{\lambda}[\boldsymbol{I}]\right|=\boldsymbol{0}$$

$$\left|\frac{l^3}{48EI}\cdot\begin{bmatrix}16 & 5\\ 5 & 2\end{bmatrix}\cdot m\cdot\begin{bmatrix}3 & 0\\ 0 & 8\end{bmatrix}-\lambda\cdot\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right|=0$$

$$\left|\begin{bmatrix}48 & 40\\ 15 & 16\end{bmatrix}-\lambda\cdot\frac{48EI}{l^3\cdot m}\cdot\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right|=0$$

$$\left|\begin{bmatrix}48 & 40\\ 15 & 16\end{bmatrix}-\alpha\cdot\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right|=0,\qquad where\ \alpha=\lambda\cdot\frac{48EI}{l^3\cdot m}$$

$$\left|\begin{bmatrix}48-\alpha & 40\\ 15 & 16-\alpha\end{bmatrix}\right|=0,$$

$$\alpha^2-64\alpha+168=0$$

$$\alpha_1=61.2575$$
$$\alpha_2=2.7425$$

$$\alpha=\frac{1}{\omega^2}\cdot\frac{48EI}{l^3\cdot m}$$

The STIFFNESS Method

$$\left|[\boldsymbol{K}]-\boldsymbol{\omega}^2\cdot[\boldsymbol{M}]\right|=\boldsymbol{0}$$

$$\left|\frac{48EI}{7l^3}\cdot\begin{bmatrix}2 & -5\\ -5 & 16\end{bmatrix}-\omega^2\cdot m\cdot\begin{bmatrix}3 & 0\\ 0 & 8\end{bmatrix}\right|=0$$

$$\left|\begin{bmatrix}2 & -5\\ -5 & 16\end{bmatrix}-\frac{7l^3\cdot\omega^2\cdot m}{48EI}\cdot\begin{bmatrix}3 & 0\\ 0 & 8\end{bmatrix}\right|=0$$

$$\left|\begin{bmatrix}2 & -5\\ -5 & 16\end{bmatrix}-\alpha\cdot\begin{bmatrix}3 & 0\\ 0 & 8\end{bmatrix}\right|=0,\qquad where\ \alpha=\frac{7l^3\cdot\omega^2\cdot m}{48EI}$$

$$\left|\begin{bmatrix}2-3\alpha & -5\\ -5 & 16-8\alpha\end{bmatrix}\right|=0,$$

$$24\alpha^2-64\alpha+7=0$$

$$\alpha_1=0.1143$$
$$\alpha_2=2.5524$$

- ***The Dynamic Characteristics***

The FLEXIBILITY Method	***The STIFFNESS Method***

The pulsation of the system

$$\omega = \sqrt{\frac{1}{\alpha} \cdot \frac{48EI}{l^3 \cdot m}} = \frac{6.928}{\sqrt{\alpha}} \cdot \sqrt{\frac{EI}{l^3 \cdot m}} \qquad [rad/s]$$

$$\omega = \sqrt{\frac{\alpha \cdot 48EI}{7l^3 \cdot m}} = 2.6186 \cdot \sqrt{\alpha} \cdot \sqrt{\frac{48EI}{l^3 \cdot m}} \quad [rad/s]$$

The pulsation of the system in the 1st **mode of vibration**

For $\alpha_1 = 61.26$,

$$\omega_1 = 0.885\sqrt{\frac{EI}{l^3 \cdot m}} = 0.885\sqrt{\frac{21 \cdot 10^{10} \cdot 19 \cdot 10^{-8}}{2^3 \cdot 10}}$$

$$\boldsymbol{\omega_1 = 19.76\ [rad/s]}$$

For $\alpha_1 = 0.1143$,

$$\omega_1 = 0.885\sqrt{\frac{EI}{l^3 \cdot m}} = 0.885\sqrt{\frac{21 \cdot 10^{10} \cdot 19 \cdot 10^{-8}}{2^3 \cdot 10}}$$

$$\boldsymbol{\omega_1 = 19.76\ [rad/s]}$$

The pulsation of the system in the **2nd mode of vibration**

For $\alpha_2 = 2.74$, $\quad \omega_2 = 4.184\sqrt{\frac{EI}{l^3 \cdot m}}$

$$\boldsymbol{\omega_2 = 93.44\ [rad/s]}$$

For $\alpha_2 = 2.5524$ $\quad \omega_2 = 4.184\sqrt{\frac{EI}{l^3 \cdot m}}$

$$\boldsymbol{\omega_2 = 93.44\ [rad/s]}$$

- *The Dynamic Characteristics*

The FLEXIBILITY Method | *The STIFFNESS Method*

The period and the frequency of the system

$$\boldsymbol{T_k = \frac{2\pi}{\omega_k}} \quad [s] \quad (12) \qquad \boldsymbol{f_k = \frac{1}{T_k}} \quad [Hz] \quad (13)$$

The period and the frequency of the system in the **1st mode of vibration**

$$\boldsymbol{T_1 = \frac{2\pi}{\omega_1}} \qquad \boldsymbol{T_1 = 0.318} \ [s]$$
$$\boldsymbol{f_1 = \frac{1}{T_1}} \qquad \boldsymbol{f_1 = 3.144} \ [Hz]$$

The period and the frequency of the system in the **2nd mode of vibration**

$$\boldsymbol{T_2 = \frac{2\pi}{\omega_2}} \qquad \boldsymbol{T_2 = 0.067} \ [s]$$
$$\boldsymbol{f_2 = \frac{1}{T_2}} \qquad \boldsymbol{f_2 = 14.87} \ [Hz]$$

- *The Dynamic Characteristics*

The FLEXIBILITY Method	*The STIFFNESS Method*

The Eigenvectors

The FLEXIBILITY Method

$$|[\Delta]\cdot[M]-\lambda[I]|\cdot\{\phi_{jk}\}=0$$

$$\left|\begin{bmatrix}48-\alpha & 40\\ 15 & 16-\alpha\end{bmatrix}\right|\cdot\begin{Bmatrix}\phi_{1k}\\ \phi_{2k}\end{Bmatrix}=0$$

$$(48-\alpha)\cdot\phi_{1k}+40\cdot\phi_{2k}=0$$
$$15\cdot\phi_{1k}+(16-\alpha)\cdot\phi_{2k}=0$$

The STIFFNESS Method

$$|[\mathrm{K}]-\omega^2\cdot[\mathrm{M}]|\cdot\{\phi_{\mathrm{jk}}\}=0$$

$$\left|\begin{bmatrix}2-3\alpha & -5\\ -5 & 16-8\alpha\end{bmatrix}\right|\cdot\begin{Bmatrix}\phi_{1k}\\ \phi_{2k}\end{Bmatrix}=0$$

$$(2-3\alpha)\cdot\phi_{1k}-5\cdot\phi_{2k}=0$$
$$-5\cdot\phi_{1k}+(16-8\alpha)\cdot\phi_{2k}=0$$

In the **1st mode of vibration**

The FLEXIBILITY Method

$$(48-\alpha_1)\cdot\phi_{11}+40\cdot\phi_{21}=0$$
$$(48-61.2675)\cdot\phi_{11}+40\cdot\phi_{21}=0$$
$$\boldsymbol{\phi_{11}=1 \quad\rightarrow\quad \phi_{21}=0.3315}$$

The STIFFNESS Method

$$(2-3\alpha_1)\cdot\phi_{11}-5\cdot\phi_{21}=0$$
$$(2-3\cdot 0.1143)\cdot\phi_{11}-5\cdot\phi_{21}=0$$
$$\boldsymbol{\phi_{11}=1 \quad\rightarrow\quad \phi_{21}=0.3315}$$

In the **2nd mode of vibration**

The FLEXIBILITY Method

$$15\cdot\phi_{12}+(16-\alpha_2)\cdot\phi_{22}=0$$
$$15\cdot\phi_{12}+(16-2.7425)\cdot\phi_{22}=0$$
$$\boldsymbol{\phi_{22}=1 \quad\rightarrow\quad \phi_{12}=-0.8838}$$

The STIFFNESS Method

$$-5\cdot\phi_{1k}+(16-8\alpha_2)\cdot\phi_{2k}=0$$
$$-5\cdot\phi_{12}+(16-8\cdot 2.5524)\cdot\phi_{22}=0$$
$$\boldsymbol{\phi_{22}=1 \quad\rightarrow\quad \phi_{12}=-0.8838}$$

- *The Dynamic Characteristics*

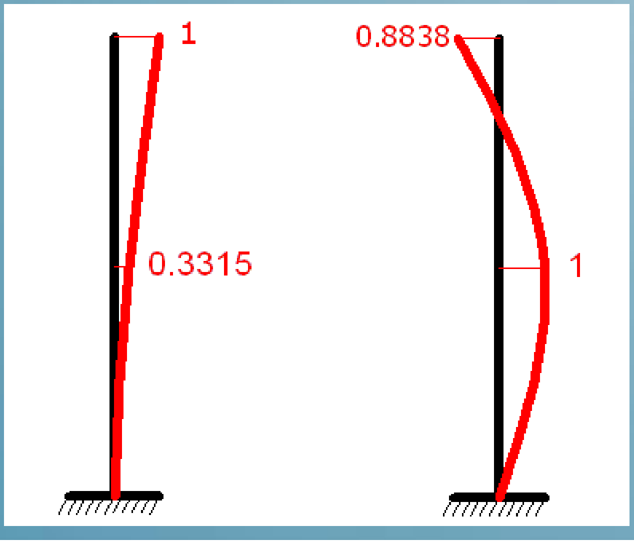
The verification of the Eigenvectors

$$\sum \boldsymbol{m_i \cdot \phi_{i,1} \cdot \phi_{i,2} = 0} \qquad (14)$$

$$m_1 \cdot \phi_{11} \cdot \phi_{12} + m_2 \cdot \phi_{21} \cdot \phi_{22} = 0$$
$$3m \cdot 1 \cdot (-0.8838) + 8m \cdot 0.3315 \cdot 1 = 0$$
$$-2.652m + 2.652m = 0$$
$$0 = 0 \ \checkmark$$

The Mode Shape Matrix

$$[\Phi] = \begin{bmatrix} 1 & -0.8838 \\ 0.3315 & 1 \end{bmatrix}$$



Modal Participation Factor

$$\boldsymbol{L_k} = \sum \boldsymbol{m_i \cdot \phi_{ki}} \quad [\boldsymbol{kg}] \qquad (15)$$

1st vibration mode:

$$L_1 = m_1 \cdot \phi_{11} + m_2 \cdot \phi_{12}$$
$$L_1 = 3m \cdot 1 + 8m \cdot (0.3315)$$
$$L_1 = 56.52 \quad [kg]$$

2nd vibration mode:

$$L_2 = m_1 \cdot \phi_{21} + m_2 \cdot \phi_{22}$$
$$L_2 = 3m \cdot (-0.8838) + 8m \cdot 1$$
$$L_2 = 53.48 \quad [kg]$$

Modal Mass

$$\boldsymbol{M_k} = \sum \boldsymbol{m_i \cdot \phi_{ki}}^2 \quad [\boldsymbol{kg}] \qquad (16)$$

1st vibration mode:

$$M_1 = m_1 \cdot \phi_{11}{}^2 + m_2 \cdot \phi_{12}{}^2$$
$$M_1 = 3m \cdot 1^2 + 8m \cdot 0.3315^2$$
$$M_1 = 38.79 \quad [kg]$$

2nd vibration mode:

$$M_2 = m_1 \cdot \phi_{21}{}^2 + m_2 \cdot \phi_{22}{}^2$$
$$M_2 = 3m \cdot (-0.8838)^2 + 8m \cdot 1^2$$
$$M_2 = 103.44 \quad [kg]$$

- *The Dynamic Characteristics*

Effective Modal Mass

$$\boldsymbol{m_{eff,k} = \frac{{L_k}^2}{M_k} \quad [kg]} \qquad (17)$$

1st vibration mode:

$$m_{eff,1} = \frac{{L_1}^2}{M_1}$$

$$m_{eff,1} = \frac{(56.52)^2}{38.79} = 82.35 \quad [kg]$$

2nd vibration mode:

$$m_{eff,2} = \frac{{L_2}^2}{M_2}$$

$$m_{eff,2} = \frac{(53.48)^2}{103.44} = 27.65 \quad [kg]$$

$$m_{eff,1} + m_{eff,2} = 82.35 + 27.65 = 110 \quad [kg]$$

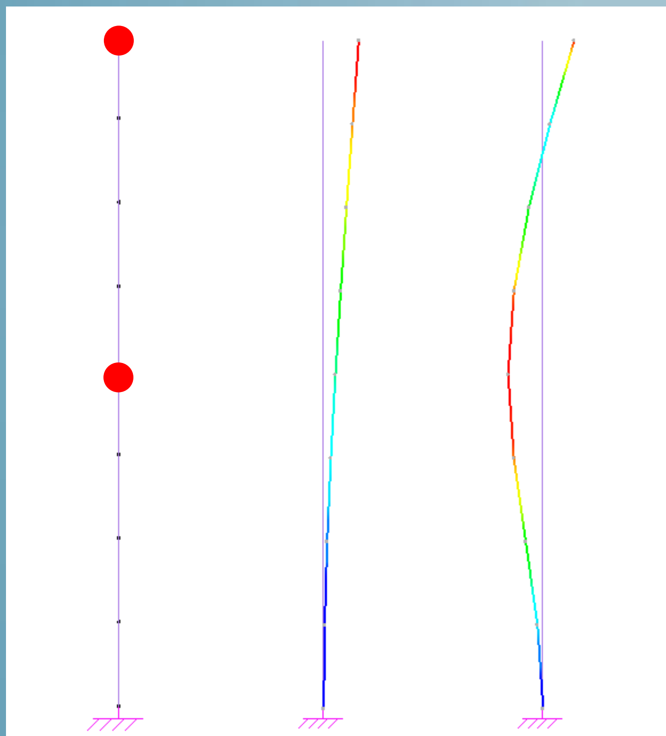
Sum of the effective masses equals the total system mass.

RESULTS

Eigen mode	Pulsation [rad/s]	Period [s]	Frequency [Hz]	Modal Mass [kg]	Effective Modal Mass [kg]
1	19.76	0.318	3.144	38.79	82.35
2	93.44	0.067	14.87	103.44	27.65

- *Modeling a 2DOF System with FEM*

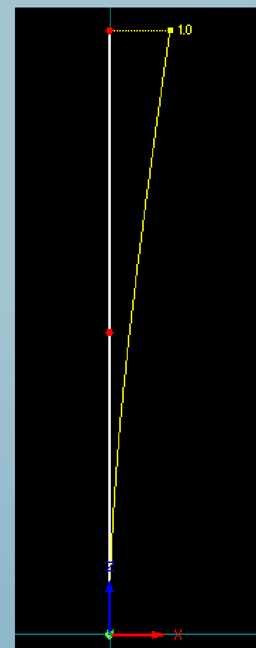
Graitec – Advance Design

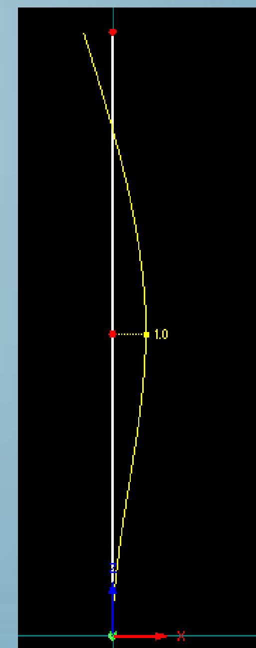


Modal analysis

	Eigenvalues				
Mode N°	Pulsation (Rad/s)	Period (s)	Frequency (Hz)	Energy (J)	Modal masses X kg (%)
1	19.71	0.32	3.14	194.24	82.54 (75.04)
2	91.46	0.07	14.56	4182.75	27.46 (24.96)
residual					0.00 (0.00)
Total				**4377.00**	**110.00 (100.00)**

Dlubal - RFEM





2.2 Eigenvectors by Node

	A	B
Node No.	E-vector No.	Standa ux [-]
1	1	0.00000
	2	0.00000
2	1	1.00000
	2	-0.88384
3	1	0.33145
	2	1.00000

	A	B	C	D
E-vector No.	Eigenvalue λ [1/s²]	Angular Frequency ω [rad/s]	Eigenfrequency f [Hz]	Eigenperiod T [s]
1	390.803	19.769	3.146	0.318
2	8728.158	93.425	14.869	0.067

	A	B	C	D	E	H
E-vector No.	Modal Mass	Participation Factor			Equivalent Mass	
	M_i [kg]	L_{iX} [kg]	L_{iY} [kg]	L_{iZ} [kg]	m_{eX} [kg]	f_{meX} [-]
1	38.79	56.52	0.00	0.00	82.34	0.749
2	103.44	53.48	0.00	0.00	27.66	0.251
Sum					110.00	1.000

• *COMPARING THE RESULTS*

	Eigen mode	Node	Eigenvectors	Pulsation	Period	Frequency	Modal Part. Factor	Modal Mass	Effective Modal Mass		
				[rad/s]	[s]	[Hz]	[kg]	[kg]	[kg]	[%]	[%]
ANALYTIC	1	1	1	19.76	0.318	3.144	56.52	38.79	82.35	74.9	100
		2	0.3315								
	2	1	-0.8838	93.44	0.067	14.87	53.48	103.44	27.65	25.1	
		2	1								
GRAITEC	1	1	0.1604	19.71	0.32	3.14	-	-	82.54	75.04	100
		2	0.05342								
	2	1	0.08724	91.46	0.07	14.56	-	-	27.46	24.96	
		2	-0.09821								
DLUBAL	1	1	1	19.769	0.318	3.146	56.52	38.79	82.34	74.9	100
		2	0.33145								
	2	1	-0.88384	93.425	0.067	14.869	53.48	103.44	27.66	25.1	
		2	1								

The agreement between the analytical (theoretical) method and the results of the industrial FEM packages is as expected very good.